

Linear Algebra (*Math 214*) Common Topics List¹

1. Computations and Proofs

- (a) The course (lectures, homework, exams) should include computations and proofs

2. Geometry and Algebra of Vectors

- (a) vector addition
- (b) scalar multiplication
- (c) linear combinations

3. Dot Product

- (a) length (norm) of vectors
- (b) unit vectors and normalizing
- (c) angle between vectors
- (d) projecting onto a vector
- (e) Cauchy-Schwarz Inequality, i.e. $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$
- (f) Triangle Inequality
- (g) orthogonal vectors

4. Lines and Planes

- (a) vector form, general form of equation of line; parametric equations
- (b) vector form, normal form, general form of equation of a plane

5. Solving Systems of Linear Equations

- (a) linear equations vs. nonlinear equations
- (b) coefficient matrix, augmented matrix
- (c) row echelon form, reduced row echelon form
- (d) Gaussian Elimination, Gauss-Jordan Elimination
- (e) leading variables, free variables
- (f) elementary row operations, row equivalent matrices
- (g) homogeneous system vs. nonhomogeneous

6. Spanning Sets and Linear Independence

- (a) the span of a set of vectors
- (b) linearly dependent, linearly independent

¹This list was approved at the 8/28/09 Math Departmental retreat

7. Matrix Operations

- (a) identity matrix, zero matrix
- (b) matrix sum, matrix product, scalar multiplication
- (c) transpose of a matrix
- (d) symmetric matrices

8. Matrix Algebra

- (a) matrix multiplication is noncommutative
- (b) associativity, left and right distributivity, multiplicative identity
- (c) properties of the transpose (e.g., $(AB)^T = B^T A^T$)

9. Inverse of a Matrix

- (a) definition of inverse and invertible
- (b) properties of invertible matrices (e.g., $(AB)^{-1} = B^{-1}A^{-1}$)
- (c) elementary matrices
- (d) Equivalent conditions for when a matrix is invertible (non-singular)

10. Subspaces, Basis, Dimension, and Rank

- (a) definition of subspace (in \mathbb{R}^n)
- (b) subspace spanned by a set of vectors
- (c) row space of A , column space of A , null space of A
- (d) rank of A , nullity of A
- (e) basis of a subspace
- (f) dimension of a subspace
- (g) standard basis of \mathbb{R}^n
- (h) finding a basis for row space, column space, and null space of a matrix
- (i) For $A_{m \times n}$, $\text{rank}(A) + \text{nullity}(A) = n$
- (j) $A\vec{x} = \vec{b}$ has a solution iff \vec{b} is in column space of A .

11. Determinants

- (a) computing determinants (expanding along rows or columns; cofactors)
- (b) how elementary row operations affect the determinant
- (c) A is singular $\iff \det(A) = 0$
- (d) theorems about determinants (e.g., $\det(AB) = \det(A)\det(B)$)

12. Eigenvalues and Eigenvectors

- (a) definitions of eigenvalues and eigenvectors
- (b) characteristic polynomial & characteristic equation
- (c) the eigenvalues are the roots of the characteristic polynomial
- (d) theorems about eigenvalues, like:
 - i. $\{\text{eigenvalues of } A\} = \{\text{eigenvalues of } A^T\}$ (b/c same char. poly.)
 - ii. If A is nonsingular (so that A^{-1} exists), then
 - λ is an eigenvalue of $A \iff \lambda^{-1}$ is an eigenvalue of A^{-1} (use same eigenvector)
 - iii. λ is an eigenvalue of $A \iff \lambda - c$ is an eigenvalue of $A - cI$ (use same eigenvector)
- (e) A is singular $\iff 0$ is an eigenvalue of A
- (f) Diagonalizable (definition)
- (g) If A has *distinct* eigenvalues, then A is diagonalizable.
- (h) $\det(A) = \prod \lambda_i$ (the determinant is the product of the eigenvalues)

13. Similarity

- (a) similar matrices
- (b) if A and B are similar, then they have the same characteristic polynomial, and therefore the same eigenvalues (& other such theorems)

14. Projections onto Subspaces and Orthogonal Complements

- (a) orthogonal complement of a subspace
- (b) Properties of W^\perp like:
 - i. if W is a subspace, W^\perp is also a subspace
 - ii. $(W^\perp)^\perp = W$
- (c) $\text{null}(A)$ is the orthogonal complement of $\text{row}(A)$, i.e., $(\text{row}(A))^\perp = \text{null}(A)$
- (d) given a vector \vec{b} and a subspace W , find \vec{b}_W and \vec{b}_{W^\perp} (where \vec{b}_W is the projection of \vec{b} onto W)
 - i. be aware that $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ and that this “decomposition” is *unique*

15. Least Squares Approximation

16. Orthogonal Bases

- (a) orthogonal set of vectors (orthogonal basis)
- (b) orthonormal basis
- (c) Gram-Schmidt process

Optional Topics for Linear Systems Math 214

- **Orthogonal Matrices and Orthogonal Diagonalization of Symmetric Real Matrices**
 - definition: $A^T A = I$
 - properties of orthogonal matrices (e.g., the columns form an *orthonormal* basis for \mathbb{R}^n (if $A_{n \times n}$), etc.)
 - If A is orthogonal then $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$, etc.
 - finding an orthogonal diagonalization of a real symmetric matrix
 - Spectral theorem (about symmetric matrices)
- **Abstract Vector Spaces and Linear Transformations**
- **LU Factorization**
- **QR Factorization**
- **Matrix Block Multiplication**
- **Cramer's Rule**
- **Gram-Schmidt Process**
- **Markov Chains**
- **Iterative Methods**
- **Singular Value Decomposition**
- **Cayley-Hamilton Theorem**